Lesson 11 Improper Integrals

Quiz Wednesday - Partial Fractions

I. FTC

Fundamental Thm of Calculus

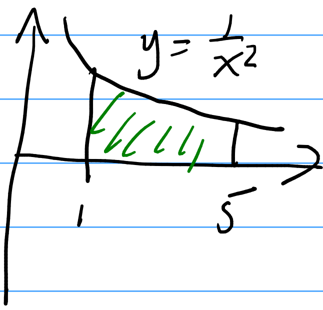
(1) If f is cont's on $[a,b]$ and $F(x)$ is any antiderivative of f then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Observations (1) a,b finite #'s
(2) f is cont's on $[a,b]$

II. Improper integrals w/ ∞ or $-\infty$.

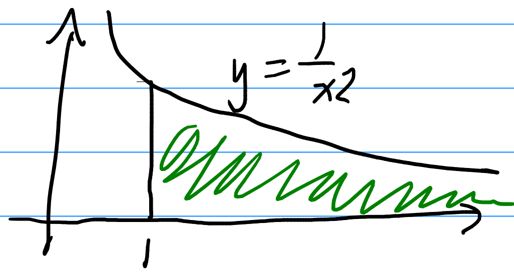
Ex) Find the area of the shaded region



$$A = \int_1^5 \underbrace{\frac{1}{x^2}}_{x^{-2}}\,dx = \frac{x^{-1}}{-1}\Big|_1^5$$

$$= \frac{-1}{x}\Big|_1^5$$

$$= \frac{-1}{5} + \frac{1}{1} = \frac{4}{5}$$



$$A = \int_1^\infty \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \frac{-1}{x}\Big|_1^b$$

$$= \lim_{b\to\infty} \cancel{-\frac{1}{b}}^{\,0} + 1$$

$$= 0 + 1$$

$$= 1$$

(D) If f is cont$\bar{s}$ on $[a, \infty)$ then

$$\int_a^{\infty} f(x)\,dx \;\underline{\underline{\text{def'n}}}\; \lim_{N\to\infty} \int_a^N f(x)\,dx$$

(Right hand side)

provided the limit on the RHS exists. If the limit does not exist, we say

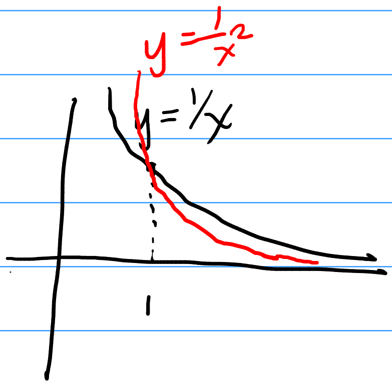
$$\int_a^{\infty} f(x)\,dx \text{ diverges.}$$

Q: How would you define $\int_{-\infty}^{a} f(x)\,dx$?

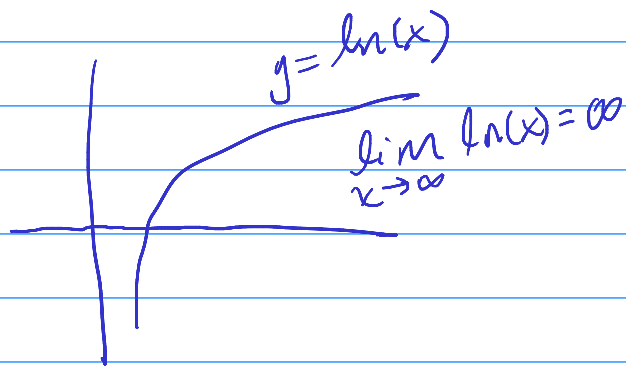
$\boxed{\text{EX}}$ $$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{N\to\infty}\int_1^N \frac{1}{x}\,dx = \lim_{N\to\infty} \ln(|x|)\Big|_1^N$$

$$= \lim_{N\to\infty} \underbrace{\ln(|N|)}_{\infty} - \underbrace{\ln(|1|)}_{0}$$

$$= \infty$$



$N \to \infty \Rightarrow |N| \to \infty$
$\Rightarrow \ln(|N|) \to \infty$



EX $\displaystyle\int_{-\infty}^{0} x^3 e^{-x^4}\,dx$

$$\int x^3 e^{-x^4}\,dx = \int e^u\left(\tfrac{-1}{4}\right)du = -\tfrac{1}{4}e^u + C$$

$u = -x^4$ $\qquad\qquad = \tfrac{-1}{4}e^{-x^4} + C$

$du = -4x^3\,dx$

$\tfrac{-1}{4}du = x^3\,dx$

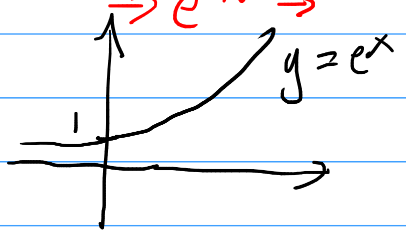
$$\int_{-\infty}^{0} x^3 e^{-x^4}\,dx = \lim_{N\to-\infty}\int_{N}^{0} x^3 e^{-x^4}\,dx = \lim_{N\to-\infty} \left.\tfrac{-1}{4}e^{-x^4}\right|_{N}^{0}$$

$$= \lim_{N\to-\infty} \underbrace{\tfrac{-1}{4}e^{0}}_{=\,\frac{-1}{4}} + \tfrac{1}{4}\underbrace{e^{-N^4}}_{\to\,0}$$

$N \to -\infty \Rightarrow N^4 \to \infty$

$\Rightarrow -N^4 \to -\infty$

$\Rightarrow e^{-N^4} \to$



$$= \frac{-1}{4}$$

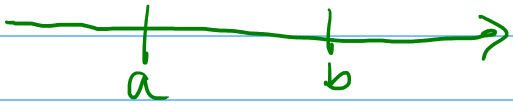
III. Improper Integrals with discontinuities

(D) If f is cont̄s on $(a,b]$ then

$$\int_a^b f(x)\,dx \overset{\text{Def'n}}{=} \lim_{N\to a^+} \int_N^b f(x)\,dx$$

If f is cont̄s on $[a,b)$ then

$$\int_a^b f(x)\,dx \overset{\text{Def'n}}{=} \lim_{N\to b^-} \int_a^N f(x)\,dx$$



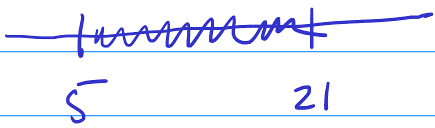
$\boxed{\text{EX}}$ $$\int_5^{21} \frac{1}{\sqrt[4]{x-5}}\,dx = \lim_{N\to 5^+} \int_N^{21} (x-5)^{-1/4}\,dx$$

discontinuity @ 5

b/c $x=5$ puts a 0 in denominator.

$$= \lim_{N\to 5^+} \frac{4}{3}(x-5)^{3/4}\Big|_5^{21}$$

$$= \lim_{N\to 5^+} \underbrace{\frac{4}{3}(16)^{3/4}}_{=\frac{4}{3}\cdot 8} - \frac{4}{3}(\underbrace{N-5}_{0})^{3/4}$$



$$= \frac{32}{3}$$

You try it

① $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{N\to 0^+} \int_N^1 \frac{1}{\sqrt{x}}\,dx$$

Ans: 2

(2) $$\int_2^\infty \frac{1}{x\ln(x)}\,dx$$

diverges